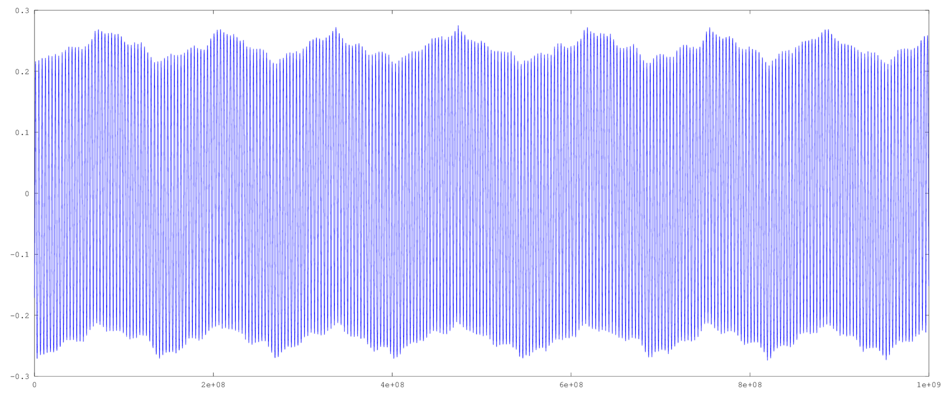

Symplectic Maps for the N-Body Problem

James Folberth
Advisor: Dr. Finn

Rose-Hulman Institute of Technology

Introduction

The N-body problem is the problem of predicting the motion of N point masses that interact gravitationally. By casting the problem in the framework of Hamiltonian mechanics, numerical integration methods can be derived that have many desirable properties. One of these methods was developed by Wisdom and Holman to predict the orbits of the outer planets for one billion years [2].

Newtonian Mechanics

Study forces on particles:

$$m_i \ddot{\mathbf{x}}_i = \mathbf{F}_i(\mathbf{x}_i, \dot{\mathbf{x}}_i, t).$$

Equations of motion are second order ODE that describe the motion of the particles $\mathbf{x}_i$. Particles have positions in the n-dimensional *configuration space* M^n. Conservative forces can be found from a potential:

$$\mathbf{F}_i = -\frac{\partial}{\partial \mathbf{x}_i} V(\mathbf{x}_1, ..., \mathbf{x}_n).$$

The N-body potential is

$$V(\mathbf{x}_1, ..., \mathbf{x}_n) = -\sum_{i<j} \frac{G m_i m_j}{r_{ij}},$$

where $r_{ij} = ||\mathbf{x}_i - \mathbf{x}_j||_2$.

Differential Forms

In coordinates x^i, a vector $\boldsymbol{\xi}$ is given by components ξ_i. A differential 1-form $\omega^1 = \sum_{j=1}^n a_j dx^j$ acts on $\boldsymbol{\xi}$ and returns a scalar:

$$\omega^1(\boldsymbol{\xi}) = \sum_{j=1}^n a_j dx^j(\boldsymbol{\xi}) = \sum_{j=1}^n a_j \xi^j$$

Forms can be constructed with the exterior product:

$$\alpha^1 \wedge \beta^1 = -\beta^1 \wedge \alpha^1$$

$$(\lambda_1 \alpha^1 + \lambda_2 \beta^1) \wedge \omega^1 = \lambda_1 \alpha^1 \wedge \omega^1 + \lambda_2 \beta^1 \wedge \omega^1$$

The exterior derivative of a 1-form ω^1 is

$$d\omega^1 = \sum_{j=1}^n da_j \wedge dx^j,$$

where da_j is the *total derivative* of a_j:

$$da_j = \sum_{k=1}^n \frac{\partial a_j}{\partial x^k} dx^k$$

Phase Space

A *symplectic structure* on M^{2n} is a closed, non-degenerate 2-form ω^2:

$$d\omega^2 = 0$$

$$\forall \boldsymbol{\xi} \neq \mathbf{0} \; \exists \boldsymbol{\eta} \ni \omega^2(\boldsymbol{\xi}, \boldsymbol{\eta}) \neq 0$$

The pair (M^{2n}, ω^2) is called a *symplectic manifold*. The *phase space* M^{2n} of a mechanical system is the space of generalized momenta and coordinates: $(\mathbf{p}, \mathbf{q})$. The phase space is naturally a symplectic manifold; the natural symplectic structure on the phase space is

$$\omega^2 = \sum_{j=1}^n dp_j \wedge dq^j.$$

A map $g^t : (\mathbf{p}, \mathbf{q}) \to (\mathbf{p}', \mathbf{q}')$ is *canonical* if and only if it preserves the symplectic structure ω^2:

$$\sum_{j=1}^n dp'_j \wedge dq'^j = \sum_{j=1}^n dp_j \wedge dq^j$$

Since g^t preserves ω^2, it also preserves ω^{2n}, which is the volume element of phase space.

Hamiltonian Mechanics

Hamiltonian mechanics is geometry in phase space [1]. The symplectic structure provides a natural isomorphism between tangent vectors and 1-forms; tangent vectors are naturally in a one-to-one correspondence with first-order, linear differential operators and solutions to systems of first-order ODE.

Newton's n second-order equations of motion are equivalent to Hamilton's $2n$ first-order equations of motion

$$\dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}} \qquad \dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}},$$

where $H(\mathbf{p}, \mathbf{q}) = T(\mathbf{p}, \mathbf{q}) + V(\mathbf{q})$ is the *Hamiltonian* (for autonomous, conservative forces).

Various techniques have been developed to find canonical transformations $g : (\mathbf{p}, \mathbf{q}) \to (\mathbf{p}', \mathbf{q}')$ that result in easily integrable equations of motion. These techniques are also used to find transformations that are useful for building numerical methods.

Symplectic Maps

Solutions to Hamilton's equations of motion form a *phase flow*; the symplectic structure is preserved by this flow, and the Hamiltonian H is conserved. We would like a numerical integration method that preserves the symplectic structure.

Consider the one-dimensional harmonic oscillator

$$H(p, q) = \frac{p^2}{2} + \frac{q^2}{2},$$

with exact solution moved forward Δt

$$\begin{pmatrix} p' \\ q' \end{pmatrix} = \begin{pmatrix} \cos(\Delta t) & -\sin(\Delta t) \\ \sin(\Delta t) & \cos(\Delta t) \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}.$$

This transformation preserves $\omega^2 : dp' \wedge dq' = dp \wedge dq$.

Consider forward Euler with time step Δt:

$$\begin{pmatrix} p' \\ q' \end{pmatrix} = \begin{pmatrix} 1 & -\Delta t \\ \Delta t & 1 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}$$

Now we have $dp' \wedge dq' = (1 + (\Delta t)^2) dp \wedge dq$. This transformation is not canonical, and the energy grows with $(1 + (\Delta t)^2)$!

N-Body Maps

In heliocentric Cartesian coordinates the N-body Hamiltonian, dominated by a large, central mass, is

$$H = \sum_{i=1}^N \frac{p_i^2}{2m_i} - \sum_{i<j} \frac{G m_i m_j}{r_{ij}}.$$

In Jacobi coordinates, the Hamiltonian is of the form

$$H = H_{Kepler} + H_{Interaction},$$

where $H_{Interaction} << H_{Kepler}$.

The mapping Hamiltonian

$$H_{Map} = H_{Kepler} + 2\pi \delta_{2\pi}(\Omega t) H_{Interaction},$$

where $\delta_{2\pi}$ is the 2π periodic δ distribution and Ω is the mapping frequency, replaces the high-frequency interaction terms with other (more manageable) high-frequency terms. The mapping operations are then evolution under only H_{Kepler} and evolution under only $H_{Interaction}$. H_{Kepler} is handled ("exactly") by standard celestial mechanics techniques; $H_{Interaction}$ is handled by symplectic integrators.

Higher-Order Methods

For a separable Hamiltonian $H(\mathbf{p}, \mathbf{q}) = T(\mathbf{p}) + V(\mathbf{q})$, a first-order *symplectic integrator* is

$$\mathbf{q}' = \mathbf{q} + \Delta t \left.\frac{\partial T}{\partial \mathbf{p}}\right|_{\mathbf{p}=\mathbf{p}} \qquad \mathbf{p}' = \mathbf{p} - \Delta t \left.\frac{\partial V}{\partial \mathbf{q}}\right|_{\mathbf{q}=\mathbf{q}'}$$

This transformation preserves the symplectic structure, but the error in $H(\mathbf{p}', \mathbf{q}')$ is only bounded by something $O(\Delta t)$.

We can build higher-order methods by composing these first-order maps in a certain way. Define

$$\mathcal{L}_H f = \left.\frac{d}{dt}\right|_{t=0} f(g_H^t(\mathbf{p}(t), \mathbf{q}(t))).$$

For some function f, the flow g_H^t advances f in time:

$$f(\mathbf{p}, \mathbf{q})\big|_{t=t_0+\Delta t} = e^{\Delta t \mathcal{L}_H} f(\mathbf{p}, \mathbf{q})\big|_{t=t_0}$$

We can take f to be a coordinate function,

$$\mathbf{q}(t + \Delta t) = e^{\Delta t \mathcal{L}_H} \mathbf{q}(t).$$

Since $H = T + V$, we have $\mathcal{L}_H = \mathcal{L}_T + \mathcal{L}_V$, and $e^{\Delta t \mathcal{L}_H} = e^{\Delta t \mathcal{L}_T + \Delta t \mathcal{L}_V}$. We seek $c_i, d_i \in \mathbb{R}$ such that

$$e^{\Delta t(\mathcal{L}_T + \mathcal{L}_V)} = \prod_{i=1}^k e^{c_i \Delta t \mathcal{L}_T} e^{d_i \Delta t \mathcal{L}_V} + o((\Delta t)^{p+1}).$$

The mapping is now the succession of mappings

$$\mathbf{q}_i = \mathbf{q}_{i-1} + c_i \Delta t \left.\frac{\partial T}{\partial \mathbf{p}}\right|_{\mathbf{p}=\mathbf{p}_{i-1}} \qquad \mathbf{p}_i = \mathbf{p}_{i-1} - d_i \Delta t \left.\frac{\partial V}{\partial \mathbf{q}}\right|_{\mathbf{q}=\mathbf{q}_i}$$

A second order method is $c_1 = c_2 = 1/2$, $d_1 = 0$, $d_2 = 1$; this method is used in the N-body simulation.

References

[1] V.I. Arnold. Mathematical Methods of Classical Mechanics
[2] J. Wisdom, M. Holman. Symplectic Maps for the N-Body Problem