Numerical Estimation of the Largest Lyapunov Exponent of Pluto

James Folberth

2 May, 2014

1 Introduction

In the late 1980s and early 1990s, considerable attention was given to the long-term behavior of the Solar System. Notably, in 1986 Applegate et al. built a special-purpose computer, the Digital Orrery, to integrate the orbits of the outer planets and Pluto for 200 million years using [1]. They also investigated the Fourier spectrum of various orbital elements of Pluto and bounded the largest Lyapunov exponent of Pluto below $10^{-6.8}\,\mathrm{yr}^{-1}$. Due to the short length of integration, they did not see convergence in their estimation of the Lyapunov exponent.

Using the Digital Orrery in 1988, Sussman and Wisdom integrated the orbits of the outer planets for 845 million years [10]. After about 300 million years, they began to see convergence in their calculation of the largest Lyapunov exponent; their reported result is $10^{-7.3}\,\mathrm{yr}^{-1}$, which corresponds to a Lyapunov timescale of about 20 million years.

Typically, a positive Lyapunov exponent is a concomitant of a chaotic dynamical system. Indeed, Sussman and Wisdom conclude that there is evidence that the orbit of Pluto is chaotic. They are hesitant to make any strong conclusions, however, due to their lack of certainty in the initial conditions and masses of the planets and Sun.

In this report, we use a symplectic integrator developed by Wisdom and Holman to estimate the largest Lyapunov exponent of Pluto [12]. The hope is that by using an entirely different integration method, using slightly different masses, and modeling Pluto as a massive particle, we can provide another estimate of the largest Lyapunov exponent. We begin by describing briefly the symplectic integrator, then the method we use to estimate the largest Lyapunov exponent, and finally we present our results.

2 The Symplectic Integrator

2.1 Why Another Integrator?

To integrate the orbits of the outer planets, the Digital Orrery used 64-bit single precision arithmetic (with certain operations in double precision) and a twelfth-order Störmer predictor with a carefully chosen time step. The time step was taken to be slightly less than 40 days; through various numerical experiments, it was found that this time step reduced the growth in the relative energy error to nearly linear growth with a very small coefficient. Over the integration, the accumulated relative energy error was on the order of 10^{-10}. However, such a low error comes at the cost of the relatively small time step and high computational cost.

Wisdom and Holman generalized one of Wisdom's previous results [11] that allows for, among other things, a much larger time step, but at the cost of a larger energy error and with the restriction of a large central mass; however, the energy error will remain within a constant bound (i.e. ignoring accumulated round-off errors, the accumulated error will not exhibit secular growth) [12, 5]. The property of bounded energy error is shared by a broad class of methods known as symplectic integrators. Symplectic integrators

are of great utility for the study of the long term behavior of Hamiltonian systems due to the bound on energy errors. Other methods, such as certain Runge-Kutta methods, do not exhibit this property, and so are not suitable for long-term integrations.

In theory, we would like an integrator that preserves both the symplectic structure of Hamiltonian system *and* preserves the Hamiltonian. However, satisfying both of these properties amounts to finding a method that gives the exact flow of the system (perhaps up to a reparameterization of time) [4]. Thus, we choose an integrator that preserves the symplectic structure of the system, but not necessarily the Hamiltonian. Given a symplectic integrator for some system, there exists a perturbed Hamiltonian for which the symplectic integrator gives the *exact* flow of the system. This result is crucial to the proof that the energy error introduced by the symplectic integrator remains bounded during the integration [5].

2.2 Wisdom and Holman's Mapping Method

Let us now move to the method developed by Wisdom and Holman [12]. Consider the Sun, the outer planets, and Pluto over a long time interval (e.g. millions of years). The orbital period of Jupiter is roughly six times larger than that of Mars. Over a long period of time, we may consider the orbits of the inner planets as inducing high-frequency perturbations in the orbits of the outer planets and Pluto. By the averaging principle, which effectively states that over a long period of time highly oscillatory perturbations tend to average to zero, we may replace the true high-frequency perturbations with other, synthetic high-frequency terms. Since the new terms will average to zero over a long period, we will not have changed the Hamiltonian too drastically.

Wisdom and Holman sought to decompose the Hamiltonian for the n-body problem into a Kepler component and an interaction component that is small relative to the Kepler component. If the Kepler component were isolated somehow, one could integrate directly the Kepler problem. Wisdom and Holman constructed a sequence of new high-frequency terms that, loosely speaking, sum to a periodic Dirac delta function. Their motivation was that when the delta function is "off", we have only Kepler components of the Hamiltonian; when the delta function is "on", we have only the interaction terms.

For the sake of brevity, we will omit some details in the description of the symplectic integrator; a proper explanation is given by Wisdom and Holman in [12]. Let $\mathbf{p}_i$ and $\mathbf{q}_i$ be the Heliocentric momentum and position vectors, respectively, of the ith orbiting body in our Solar system. Denoting the masses of the orbiting bodies by m_i, the n-body Hamiltonian is

$$H = \sum_{i=0}^{n-1} \frac{\|\mathbf{p}_i\|^2}{2m_i} - \sum_{i<j} \frac{Gm_i m_j}{r_{ij}}, \tag{1}$$

where $r_{ij} = \|\mathbf{q}_i - \mathbf{q}_j\|$. A Hamiltonian is said to be a *Kepler* Hamiltonian (or *Keplerian*) if it can be written in the form

$$H_K = \frac{\|p\|^2}{2m} - \frac{GMm}{r},$$

or as a sum of such Hamiltonians. Kepler Hamiltonians have been studied in much detail: a system with a single Kepler Hamiltonian is exactly integrable, and there are very fast and accurate numerical methods for integrating the orbit of a Keplerian system. This is key to the computational speed of Wisdom and Holman's symplectic integrator.

However, the n-body Hamiltonian (1) (in Heliocentric coordinates) is not in the form of a sum Kepler Hamiltonians plus a *small* interaction Hamiltonian. The solution to this problem is to switch to Jacobi coordinates. The Jacobi coordinates (denoted with a prime) are constructed by setting $\mathbf{q}'_0$ to be the center of mass of the n-body system. Then the first relative coordinate $\mathbf{q}'_1$ is the position of the first body relative to the central mass (the Sun). The second relative coordinate $\mathbf{q}'_2$ is the position of the second body relative

to the center of mass of the central mass and first body. This process can be generalized to the other relative coordinates. Let $\mathbf{q}'_0$ be the center of mass of the system. Then the remaining Jacobi (position) coordinates $(1 \leq i \leq n-1)$ are given by

$$\mathbf{q}'_i = \mathbf{q}_i - \mathbf{Q}_i \qquad \text{where} \quad \mathbf{Q}_i = \frac{1}{\eta_i} \sum_{j=0}^{i} m_j \mathbf{q}_j \quad \text{and} \quad \eta_i = \sum_{j=0}^{i} m_i.$$

We also have that that $\mathbf{p}'_i = m'_i \mathbf{v}'_i$, where $\mathbf{v}'_i$ is the time derivative of $\mathbf{q}'_i$ and the new mass factors are $m'_i = \eta_{i-1} m_i / \eta_i$ for $1 \leq i \leq (n-1)$ and $m'_0 = \eta_{n-1}$. Note that $\eta_{n-1} = M$ is the total mass of the system and that the transformation to Jacobi coordinates is linear.

In mixed Heliocentric and Jacobi coordinates, the n-body Hamiltonian (1) becomes (after adding and subtracting a certain quantity)

$$H = \sum_{i=1}^{n-1} \left(\frac{||\mathbf{p}'_i||^2}{2m'_i} - \frac{G m_i m_0}{r'_i} \right) + \frac{||\mathbf{p}'_0||^2}{2M} + \sum_{i=1}^{n-1} \left(\frac{G m_i m_0}{r'_i} - \frac{G m_i m_0}{r_{i0}} \right) - \sum_{0<i<j} \frac{G m_i m_j}{r_{ij}}, \tag{2}$$

where $r'_i = \|\mathbf{q}'_i\|$ and r_{ij} is as before. This Hamiltonian is, thus far, equivalent to the n-body Hamiltonian (1); we have made no approximations. Note, however, that part of the mixed-coordinate Hamiltonian (2) has $n-1$ Keplerian components. The remaining terms involve the momentum of the center of mass, subtracting nearly equal quantities, and the interaction of planets to planets (including Pluto). The last two terms are small, and so the n-body Hamiltonian is successfully decomposed into the form

$$H = H_{Kepler} + H_{Interaction},$$

where $H_{Kepler} \ll H_{Interaction}$.

With the Hamiltonian in the proper form, Wisdom and Holman invoke the averaging principle to argue that they can replace the true high-frequency terms with other high-frequency terms. They pick higher and higher frequency cosine terms that "sum" to a 2π-periodic Dirac delta function $\delta_{2\pi}$ (Dirac comb). They define the Hamiltonian for their symplectic integrator/mapping as

$$H_{SI} = H_{Kepler} + 2\pi \delta_{2\pi}(t) H_{Interaction}. \tag{3}$$

The Keplerian component, H_{Kepler}, can be integrated using variety of methods. The choice recommended by Wisdom and Holman is to use the f and g functions of orbital mechanics to integrate exactly (within roundoff errors) the motion of the bodies under the Keplerian Hamiltonian [3]. Note that the Kepler terms are in Jacobi coordinates.

The interaction component, $H_{Interaction}$, periodically "kicks" the orbits of the planets. In our implementation (Section 4), we use Ruth's second order, explicit symplectic integrator for this step [8]. While Ruth's second order method is not extremely accurate, it is sufficient for a "qualitatively accurate" integration. The integrator requires the first order derivatives of the interaction Hamiltonian; these are given in [9], along with a brief discussion of their efficient and accurate computation.

Now that we can handle separately the Keplerian and interaction Hamiltonians, we may piece them together. Simply put, we start off with one time step of integration under the Kepler Hamiltonian. At the end of the integration time step, the delta function turns "on" and we integrate under the interaction Hamiltonian for a time step to "catch up" with where the Kepler integration left off. The process of doing a time step of Kepler and then a time step of interaction forms one iteration of Wisdom and Holman's symplectic integrator, and moves the system forward in time one time step.

3 Estimation of the Largest Lyapunov Exponent

We now focus on estimating the largest Lyapunov exponent of Pluto. Due to the peculiar orbit of Pluto, there is reason to believe that Pluto's orbit has chaotic characteristics. It is therefore justifiable to attempt to estimate the largest Lyapunov exponent of the orbit of Pluto[1]. There are two standard approaches to estimate the largest Lyapunov exponent of a dynamical system. We present both methods, as the first method motivates the more accurate, second method.

3.1 First Method - Two Particles

Consider the position of Pluto in phase space at some fixed, reference initial condition. Using the symplectic integrator (or perhaps another method), we can evolve the orbit of Pluto and the other bodies in time. We call this orbit the reference orbit; in the literature, it is sometimes known as the fiducial orbit.

Now consider a "copy" of the Solar System with a nearly identical Pluto, in that everything is the same as the reference Solar System, except that the initial conditions for Pluto are *slightly* changed. On might choose an initial separation that is a few orders of magnitude larger than machine precision. By integrating the perturbed Solar System with the same method as the other system, one can determine if the two Pluto trajectories are diverging or not. For a chaotic system, we expect them to diverge (roughly) exponentially, which corresponds to a positive Lyapunov exponent.

There is a limitation to this method, however. We expect that Pluto will remain in the Solar system for the foreseeable future. Therefore, the two versions of Pluto cannot get infinitely far apart, meaning that at some point, the distance between the two versions of Pluto will "saturate". This problem can be solved (somewhat) if one integrates both trajectories for some number of time steps and records the distance between the reference Pluto and the perturbed Pluto. Then the perturbed Pluto is moved back to the reference trajectory and the integration continues, so it can begin separating from the reference trajectory. In practice, the length of time one integrates before "renormalizing" has an effect on the computed Lyapunov exponent [10]. The second method of estimating the largest Lyapunov exponent does not have this problem, so we use it for our calculations.

3.2 Second Method - Variational

The second method, one might call it the *variational method*, uses the original Hamiltonian system of ODEs and its linearization. Suppose we have Pluto with reference initial conditions x_0. Let φ_t be the Hamiltonian flow that moves x_0 along the reference trajectory. If we perturb the initial conditions of the reference trajectory by an infinitesimal amount ϵv_0, we can linearize the flow about the reference trajectory $\varphi_t(x_0)$ to determine how the perturbation $v(t)$ behaves in time; this is analogous to measuring the distance between Pluto and a perturbed copy of Pluto. Assuming that φ_t is sufficiently smooth, we have

$$\varphi_t(x_0 + \epsilon v_0) = \varphi_t(x_0) + \epsilon D_x \varphi_t(x_0) v_0 + o(\epsilon).$$

We then have that the deviation vector v evolves as $v(t) = D_x \varphi_t(x_0) v_0$. We can write Hamilton's equations as $\dot{x} = f(x)$, where f is a vector field on the phase space. Assuming that f is sufficiently smooth, we observe that the ODE for φ_t becomes

$$\frac{d}{dt}\left(\varphi_t(x_0) + \epsilon v(t)\right) = f(\varphi_t(x_0)) + \epsilon Df(\varphi_t(x_0)) v(t) + o(\epsilon).$$

[1] For experimental data and simulations like these, one may prefer to speak of Lyapunov characteristic indicators, instead of Lyapunov exponents. The sign and magnitude of the largest exponent/indicator is one measure of how chaotic a system may be.

Noting that $\frac{d}{dt}\varphi_t(x_0) = f(\varphi_t(x_0))$, we have

$$\dot{v} = Df(\varphi_t(x_0))v. \tag{4}$$

We therefore have a system of ODEs for evolution of the perturbation $v(t)$. Suppose M is the phase space. We know that $\varphi_t : M \to M$ takes points in phase space and moves them forward in time. The initial perturbation is an element of the tangent space of M at the point x_0: in notation, $v_0 \in T_{x_0}M$. Therefore, the map $Df(\varphi_t(x_0)) : T_{x_0}M \to T_{\varphi_t(x_0)}M$ takes vectors in the tangent space at x_0 and maps them to corresponding vectors in $T_{\varphi_t(x_0)}M$. By integrating the ODE for v along side integrating to find $\varphi_t(x_0)$, we are transporting v along with the flow.

Again, in a chaotic system, we expect that an infinitesimal perturbation to the initial conditions will eventually result in an exponential growth of $\|v(t)\|$. A common definition, given in [6], for the largest Lyapunov exponent μ_{max} is

$$\mu_{max} = \limsup_{t\to\infty} \frac{1}{t} \ln \|v(t)\|.$$

It is somewhat typical to have in addition that the following limit exists:

$$\lim_{t\to\infty} \frac{1}{t} \ln \|v(t)\| = \mu_{max}.$$

With the variational method, we do not (in theory) worry about the size of $\|v(t)\|$; we can let the magnitude of $v(t)$ grow without having to worry about saturation, since $v(t)$ is not the distance between two points in the phase space. In practice, however, we will run into numerical difficulties if we let $\|v(t)\|$ grow without bound. If we use an adaptive step size ODE solver for the deviation vector ODE (4), it may begin to take smaller and smaller step sizes attempting to control the error, until a minimum step size is reached and the solver stops. We may also run into numerical overflow if v gets sufficiently large.

Since the deviation vector ODE is linear, we may, at any time, normalize $v(t)$ by dividing by $\|v(t)\|$. This is analogous to moving the copy of Pluto back to the reference trajectory in the first method. We must keep track of the normalizing factors when we estimate μ_{max}. An algorithm to approximate μ_{max} is as follows [7]:

1. Let $v^{(0)} = \|v_0\|$.
2. Integrate both the reference trajectory and the variational equation for a time T.
3. Let $v^{(1)} = v^{(0)}\|v(T)\|$. Set $v(T) \leftarrow v(T)/\|v(T)\|$.
4. Repeat this integration and normalization procedure N times, keeping track of $v^{(i)}$ for $i = 0, 1, ..., N$. After N steps, we have $v^{(N)} = v^{(N-1)}\|v(NT)\|$.
5. Terminating the integration, we have

$$\mu_{max} \approx \frac{1}{NT} \ln \|v^{(N)}\| \tag{5}$$

For long time integrations, NT will be large, and for chaotic systems $v^{(N)}$ will be very large. It is recommended to keep track of $\ln \|v^{(n)}\|$ instead of $\|v^{(n)}\|$ to prevent a loss of precision. In the above procedure, we would instead add $\ln \|v(nT)\|$ to $\ln \|v^{(n-1)}\|$ when we compute $\ln \|v^{(n)}\|$. In an implementation, we would of course monitor the convergence of μ_{max}, and once it has (hopefully) settled down to a reasonable value, we would halt the computation. It is typical to view a "loglog" plot of μ_{max} versus time; see Figure 3 for an example.

4 Implementation, Results, and Discussion

Fortran 90 source code for the symplectic integrator and μ_{max} estimation is available at `https://github.com/jamesfolberth/nbody_si_fort`. The code depends on external HDF5 and ODEPACK libraries. ODEPACK is available at `http://www.netlib.org/`; HDF5 is available at `http://www.hdfgroup.org/HDF5/`. Data from the computations were written to HDF5 data files and read in to GNU Octave for analysis. Plotting was done with Gnuplot; the plots were "printed" to TikZ files and compiled with LuaLaTeX.

4.1 Implementing the Symplectic Integrator

To estimate the largest Lyapunov exponent μ_{max}, we first needed a fast implementation of Wisdom and Holman's symplectic integrator. There are two main components to accomplish this: first, a Kepler integrator, and second, a symplectic integrator to "integrate over the delta function" when the interaction Hamiltonian is "on". Also needed are various utilities to convert to and from Jacobi coordinates in an efficient manner.

The standard way of converting to Jacobi coordinates is to index bodies according to increasing semi-major axis. This will index Pluto as the last planet. Due to the structure of Jacobi coordinates, the motions of the planets interior to Pluto will be "encoded" into Pluto's Jacobi coordinates. The inner planets, notably Jupiter, will induce small amplitude, high-frequency oscillations in the Jacobi coordinates of Pluto. To avoid this, we simply index Pluto as the first planet in the conversion to Jacobi coordinates.

To convert to and from Jacobi coordinates, we first note that the transformation is linear. In our code, we represent the positions $\mathbf{q}$ and momenta $\mathbf{p}$ by $3n$ vectors, with the x, y, z components of each body stored contiguously in the vector. In this form, the transformation to Jacobi coordinates can be written as a *sparse* matrix-vector product; the inverse transformation can be implemented with a precomputed PLU matrix decomposition, and then using a sparse forward and backward substitution routine.

The Kepler integrator, which integrates in Jacobi coordinates, is taken from Danby [3]. The method computes the semi-major axis, mean motion, and eccentricity, and a few other quantities, but not a complete set of orbital elements. Kepler's equation $M = E - e \sin E$ is then solved with a quartic Halley-type method. Finally, Gauss/Lagrange's f and g functions are used to find the updated positions $\mathbf{q}'$ and momenta $\mathbf{p}'$.

The symplectic integrator for $H_{Interaction}$ is Ruth's second order, explicit method [8]. For ease of notation, take $H_{Int} = H_{Interaction}$. The integrator needs that H_{Int} is separable, meaning that $H_{Int} = T + V$, where T is a kinetic term and V is a potential term. The integrator then updates the momenta and position with terms involving $\partial T/\partial \mathbf{p}'$ and $\partial V/\partial \mathbf{q}'$; the primes indicate that these derivatives are taken with respect to Jacobi coordinates. It does not add much to the discussion, so we omitted writing derivatives here.

With the Digital Orrery, the time step was taken to be slightly less than 40 days. Through various experiments, it was found that this time step reduced the linear growth factor in the relative Hamiltonian error by about three orders of magnitude. While this result is quite impressive, 40 days is quite a small time step. Part of the efficiency gained by Wisdom and Holman's symplectic integrator is that we can take significantly larger time steps. Wisdom and Holman found that a time step approximately 1/5th that of Jupiter's orbital period worked well. We use a time step of 1 year.

The initial conditions for the Solar System were taken from [1], the first published Digital Orrery integration. Other uses of these initial conditions include [10] and [12]. The masses for the Sun, planets, and Pluto were taken from [2]. Note that the masses we use are slightly different than those in [1].

4.2 Implementing the Variational Equation Solver

The next step is to implement the solver for the variational equation. We must first find the linearized equations; we do this in Heliocentric coordinates. Let $x = \begin{bmatrix}\mathbf{q} & \mathbf{p}\end{bmatrix}^T$. We may write the Hamiltonian system of ODEs as $\dot{x} = f(x)$, where

$$f = \begin{bmatrix}\frac{\partial H}{\partial \mathbf{p}} & -\frac{\partial H}{\partial \mathbf{q}}\end{bmatrix}^T.$$

We then have that, since H is separable,

$$Df = \begin{bmatrix}\frac{\partial^2 H}{\partial \mathbf{q}\partial \mathbf{p}} & \frac{\partial^2 H}{\partial \mathbf{p}^2} \\ -\frac{\partial^2 H}{\partial \mathbf{q}^2} & -\frac{\partial^2 H}{\partial \mathbf{p}\partial \mathbf{q}}\end{bmatrix} = \begin{bmatrix}0 & \frac{1}{\mathbf{m}} \\ -\frac{\partial^2 H}{\partial \mathbf{q}^2} & 0\end{bmatrix},$$

where $\mathbf{m}$ is the diagonal matrix of masses. Computing $\partial^2 H/\partial \mathbf{q}^2$ is tedious, but straightforward; these calculations are given in Appendix A.

We start with a "random" initial perturbation of Pluto $v(0)$. Let $\|v(t)\|_P$ denote the 2-norm of the position and momentum of Pluto, treated as a single 6-vector. In the code, position is in AU and momentum in AU/year, so there is a disparity in the units; however, $\|\cdot\|_P$ should be equivalent to other norms. At each stage of the integration, we will use the norm $\|\cdot\|_P$ to normalize the perturbation $v(t)$.

The time stepping is composed of integrating the variational equations $\dot{v} = Df(x(t))v$ for one time step, and then integrating the reference trajectory with the symplectic integrator. We use a linear multi-step method to integrate the linear system (LSODE, method 10). Once we have moved forward 1000 time steps, we store the norm $\|v\|_P$, normalize v, and update our estimate of μ_{max}. The approximations of μ_{max} stored over time are used to monitor the convergence of μ_{max}.

4.3 Results and Discussion

First, we present briefly the results of integrating the reference trajectory. On a computer with a 1.6 GHz Intel CPU (Core i5), the GNU Fortran compiler (4.8.2), and modest compiler optimization, integrating the reference trajectory for 1 billion years with a time step of 1 year took approximately 70 minutes. The positions and momenta in Heliocentric and Jacobi coordinates, along with the necessary data to compute the Hamiltonian and orbital elements, were stored every 1000 time steps. The data were saved to an HDF5 data file and loaded into GNU Octave with a gnuplot graphics backend for analysis.

In Figure 1 we show the relative error in the Hamiltonian. Note that while the error is fluctuating wildly, it remains bounded for the duration of the integration. We expect exactly this behavior from a symplectic integrator [5].

In Figure 2 we show the function $h = e\sin(\omega + \Omega)$, a function of three orbital elements of Pluto, over the integration timescale. Two nearly periodic behaviors are immediate from the plot: a long 137 million year period and a short 3.7 million year period. The interested reader should compare Figure 2 to the corresponding figures in [10] and [12]; we have found only slight differences, likely due to different masses and modeling Pluto as a massive particle.

We also integrated Pluto as a massless particle, using the same initial conditions as other integrations. After only 200 million years we noticed significant differences between the orbit of a massless Pluto and a massive Pluto. The phase of the 3.7 million year oscillation in h for a massless Pluto fell about 10° degrees behind the phase for a massive Pluto[2]; the amplitude was also slightly different. Overall, however, the two

[2]Note that this is *not* to say that either the massive or massless Pluto had the true phase of Pluto.

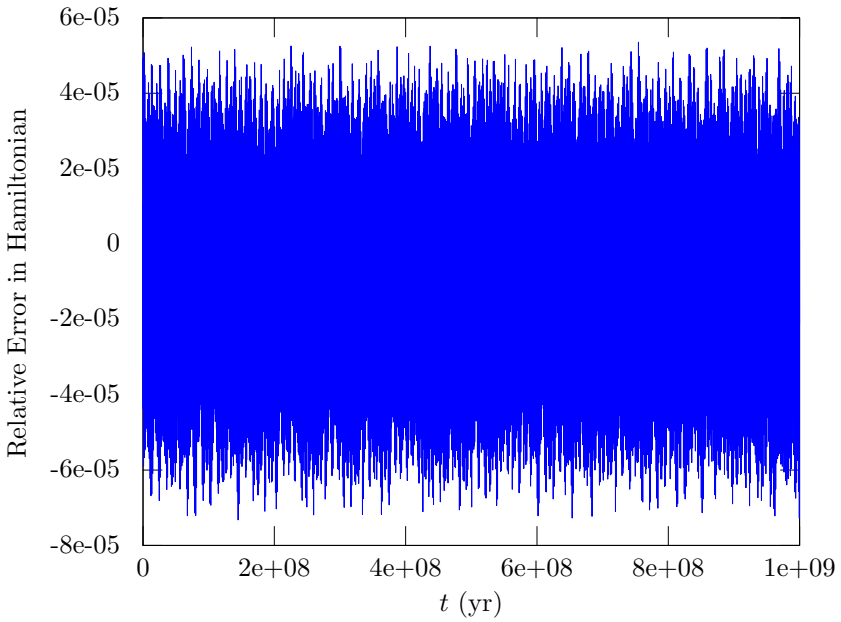

Figure 1: Relative error in Hamiltonian for reference trajectory integration.

orbits were quite similar.

Figure 3 shows the behavior of the approximations (5) of μ_{max} over almost the entire integration time scale. Occasionally, the estimate will drop well below the linear trend (e.g. near $t = 10^4$ years); these are also present in the corresponding figures from [1, 10].

Figure 4 enlarges the region from $t = 10^8$ years to $t = 10^9$ years. We see a rather sharp jump in the estimate of the exponent, but note that this occurs after only 15% of the integration time. After the jump, the estimate begins to decay nearly exponentially, and seems to converge to approximately $10^{-7.0}$ /year. Fitting a decaying exponential confirms this, and we find $\mu_{max} \approx 10^{-7.0}$ /year. This corresponds to a Lyapunov time of $\mu_{max}^{-1} \approx 10$ million years.

Our results are in agreement with the upper bound given by Applegate [1], and are comparable to the results given by Sussman and Wisdom (20 million years) [10, 12]. We suspect that the primary causes of the discrepancy between our results and Sussman and Wisdom are the slight differences in masses and the fact that we treat Pluto as a massive particle. Yet, even with these differences, we still reach the same conclusion as Sussman: Pluto appears to exhibit chaotic behavior on a short time scale, relative to the age of the Solar System.

Sussman and Wisdom note that accounting for a massive Pluto would affect the orbits of the planets; a chaotic Pluto would imply chaos in the entire Solar system [10]. Saha and Tremaine have developed a generalization of Wisdom and Holman's symplectic integrator that uses individual time steps for each planet [9]. Their method significantly increases the speed of Wisdom and Holman's method, and can be used to include the inner planets in the integration. One avenue of future work is to include the effects of the inner planets in the integration. Due to the relatively high orbital frequency of the inner planets, we expect, due to the averaging principle, that their contribution in the integration to be minimal.

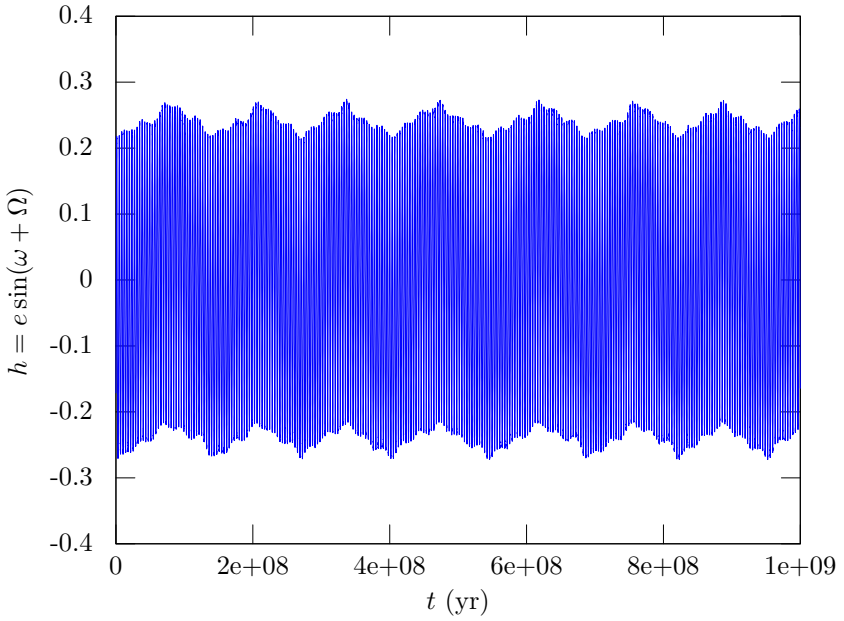

Figure 2: $h = e\sin(\omega + \Omega)$ for Pluto for reference trajectory integration.

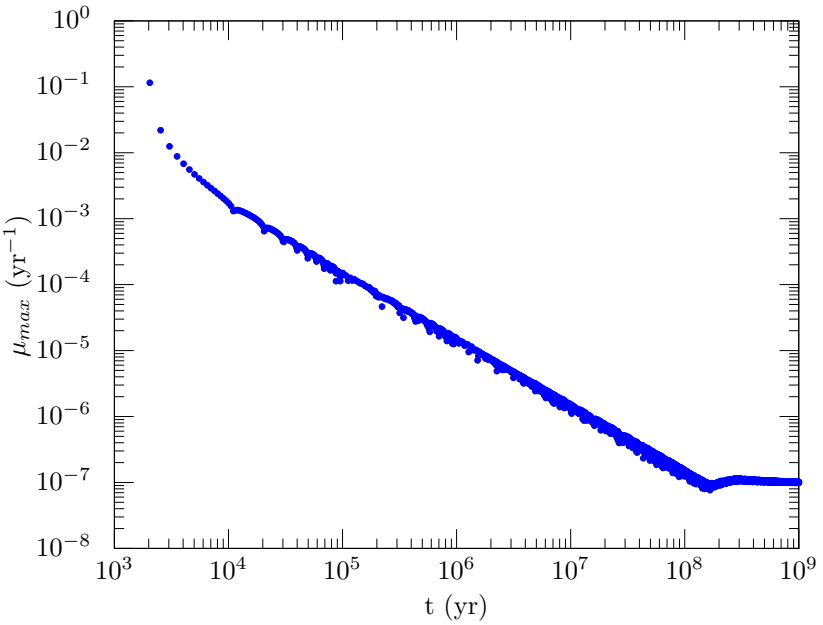

Figure 3: Convergence of largest Lyapunov exponent for massive Pluto.

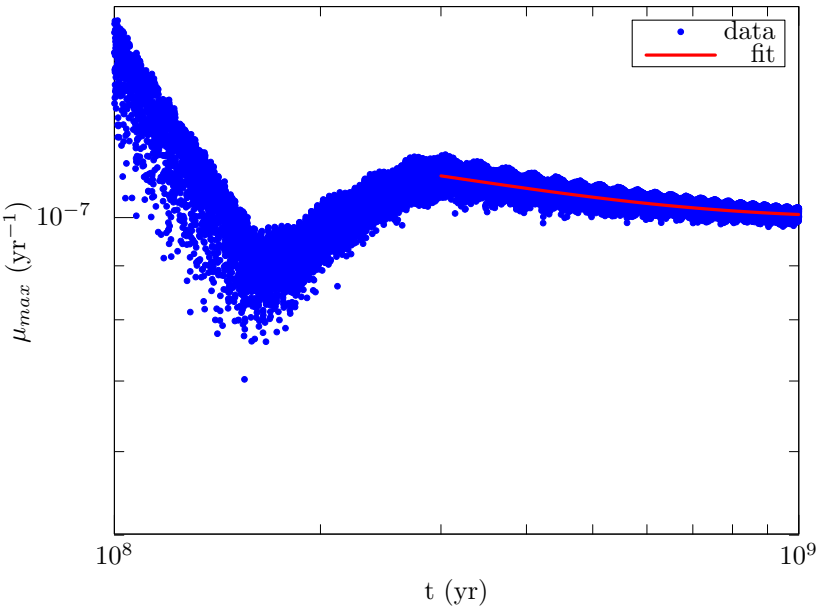

Figure 4: Exponential fit to estimate largest Lyapunov exponent.

A Various Derivatives

In this appendix we present the derivatives necessary for the variational equation. Let $\dot{x} = f(x)$ be the Hamiltonian system of ODEs with $x = \begin{bmatrix}\mathbf{q} & \mathbf{p}\end{bmatrix}^T$ and

$$f = \begin{bmatrix}\dfrac{\partial H}{\partial \mathbf{p}} & -\dfrac{\partial H}{\partial \mathbf{q}}\end{bmatrix}^T,$$

and Hamiltonian (1). Note that $\mathbf{q} = \begin{bmatrix}\mathbf{q}_1 & \cdots & \mathbf{q}_n\end{bmatrix}^T$. The only nontrivial derivatives to compute are $\partial H/\partial \mathbf{q}$ and $\partial^2 H/\partial \mathbf{q}^2$. We then have that

$$\frac{\partial H}{\partial \mathbf{q}_k} = -\sum_{i=0}^{n-1} G m_i m_k \frac{\mathbf{q}_i - \mathbf{q}_k}{r_{ik}^2} \quad i \neq k.$$

Taking a second derivative, we have

$$\frac{\partial}{\partial \mathbf{q}_m}\frac{\partial H}{\partial \mathbf{q}_k} = \frac{G m_m m_k}{r_{mk}^3}\left(\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} - \frac{3}{r_{mk}^2}(\mathbf{q}_m - \mathbf{q}_k)(\mathbf{q}_m - \mathbf{q}_k)^T\right) \quad m \neq k,$$

and

$$\frac{\partial}{\partial \mathbf{q}_k}\frac{\partial H}{\partial \mathbf{q}_k} = \sum_{\substack{i=0\\ i\neq k}}^{n-1} \frac{G m_i m_k}{r_{ik}^3}\left(\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} - \frac{3}{r_{ik}^2}(\mathbf{q}_i - \mathbf{q}_k)(\mathbf{q}_i - \mathbf{q}_k)^T\right).$$

References

[1] J. Applegate, M. Douglas, Y. Gürsel, G. Sussman, and J. Wisdom, *The outer solar system for 200 million years*, The Astronomical Journal, (1986).

[2] B. W. Carroll and D. A. Ostlie, *An Introduction to Modern Astrophysics*, Pearson, 2 ed., 2007.

[3] J. Danby, *Fundamentals of Celestial Mechanics*, Willmann-Bell, 2 ed., 2003.

[4] Z. Ge and J. Marsden, *Lie-poisson hamilton-jacobi theory and lie-poisson integrators*, Physical Letters A, (1988).

[5] E. Hairer, *Geometric Numerical Integration*, Springer, 2006.

[6] J. D. Meiss, *Differential Dynamical Systems*, SIAM, 2007.

[7] T. S. Parker and L. O. Chua, *Practical Numerical Algorithms for Chaotic Systems*, Springer, 1989.

[8] R. Ruth, *A canoncial integration technique*, IEEE Transactions on Nuclear Science, (1983).

[9] P. Saha and S. Tremaine, *Long-term planetary integration with individual time steps*, astro-ph/9403057, (1994).

[10] G. Sussman and J. Wisdom, *Numerical evidence that the motion of pluto is chaotic*, Science, (1988).

[11] J. Wisdom, *Chaotic behavior and the origin of the 3/1 kirkwood gap*, Icarus, (1983).

[12] J. Wisdom and M. Holman, *Symplectic maps for the n-body problem*, The Astronomical Journal, (1991).